

Weighted Efficient Domination for $(P_5 + kP_2)$ -Free Graphs in Polynomial Time

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Abstract

Let G be a finite undirected graph. A vertex *dominates* itself and all its neighbors in G . A vertex set D is an *efficient dominating set* (*e.d.* for short) of G if every vertex of G is dominated by exactly one vertex of D . The *Efficient Domination* (ED) problem, which asks for the existence of an e.d. in G , is known to be NP-complete even for very restricted graph classes such as for claw-free graphs, for chordal graphs and for $2P_3$ -free graphs (and thus, for P_7 -free graphs). We call a graph F a *linear forest* if F is cycle- and claw-free, i.e., its components are paths. Thus, the ED problem remains NP-complete for F -free graphs, whenever F is not a linear forest. Let WED denote the vertex-weighted version of the ED problem asking for an e.d. of minimum weight if one exists.

In this paper, we show that WED is solvable in polynomial time for $(P_5 + kP_2)$ -free graphs for every fixed k , which solves an open problem, and, using modular decomposition, we improve known time bounds for WED on $(P_4 + P_2)$ -free graphs, $(P_6, S_{1,2,2})$ -free graphs, and on $(2P_3, S_{1,2,2})$ -free graphs and simplify proofs. For F -free graphs, the only remaining open case is WED on P_6 -free graphs.

Keywords: Weighted efficient domination; F -free graphs; linear forests; P_k -free graphs; polynomial time algorithm; robust algorithm.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph. A vertex $v \in V$ *dominates* itself and its neighbors. A vertex subset $D \subseteq V$ is an *efficient dominating set* (*e.d.* for short) of G if every vertex of G is dominated by exactly one vertex in D . Note that not every graph has an e.d.; the EFFICIENT DOMINATING SET (ED) problem asks for the existence of an e.d. in a given graph G . If a vertex weight function $\omega : V \rightarrow \mathbb{N}$ is given, the WEIGHTED EFFICIENT DOMINATING SET (WED) problem asks for a minimum weight e.d. in G if there is one or for determining that G has no e.d.

For a set \mathcal{F} of graphs, a graph G is called \mathcal{F} -free if G contains no induced subgraph isomorphic to a member of \mathcal{F} . For two graphs F and G , we say that G is F -free if G is $\{F\}$ -free. We denote by $G + H$ the disjoint union of graphs G and H . Let P_k denote a chordless path with k vertices, and let $2P_k$ denote $P_k + P_k$, and correspondingly for kP_2 . The *claw* is the 4-vertex tree with three vertices of degree 1.

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Many papers have studied the complexity of ED on special graph classes - see e.g. [2] for references. In particular, ED remains NP -complete for $2P_3$ -free graphs, for chordal graphs, for line graphs and thus for claw-free graphs.

A *linear forest* is a graph whose components are paths; equivalently, it is a graph which is cycle-free and claw-free. The NP -completeness of ED on chordal graphs and on claw-free graphs implies: If F is not a linear forest, then ED is NP -complete on F -free graphs. This motivates the analysis of ED/WED on F -free graphs for linear forests F .

In this paper, we show that WED is solvable in polynomial time for $(P_5 + kP_2)$ -free graphs for every fixed k , which solves an open problem, and, using modular decomposition, we improve known time bounds for WED on $(P_4 + P_2)$ -free graphs, $(P_6, S_{1,2,2})$ -free graphs, and on $(2P_3, S_{1,2,2})$ -free graphs and simplify proofs (see [2, 5] for known results). For F -free graphs, the only remaining open case is WED on P_6 -free graphs.

Various of our algorithms are robust in the sense of [7], that is, a robust algorithm for a graph class \mathcal{C} works on every input graph G and either solves the problem correctly or states that $G \notin \mathcal{C}$. We say that the algorithm is *weakly robust* if it either gives the optimal WED solution for the input graph G or states that G has no e.d. or is not in the class.

2 Basic Notions and Results

2.1 Some Basic Notions

All graphs considered in this paper are finite, undirected and simple (i.e., without loops and multiple edges). For a graph G , let $V(G)$ or simply V denote its vertex set and $E(G)$ or simply E its edge set; throughout this paper, let $|V| = n$ and $|E| = m$. We can assume that G is connected (otherwise, WED can be solved separately for its components); thus, $m \geq n - 1$. A graph is *nontrivial* if it has at least two vertices. For a vertex $v \in V$, $N(v) = \{u \in V \mid uv \in E\}$ denotes its (*open*) *neighborhood*, and $N[v] = \{v\} \cup N(v)$ denotes its *closed neighborhood*. A vertex v *sees* the vertices in $N(v)$ and *misses* all the others. The *anti-neighborhood* of vertex v is $A(v) = V \setminus N[v]$.

For a vertex set $U \subseteq V$, its neighborhood is $N(U) = \{x \mid x \notin U, \exists y \in U, xy \in E\}$, and its anti-neighborhood $A(U)$ is the set of all vertices not in U missing U .

The *degree* of a vertex x in a graph G is $d(x) := |N(x)|$. Let $\delta(G)$ denote the minimum degree of any vertex in G .

A vertex u is *universal* for $G = (V, E)$ if $N[u] = V$. Independent sets, complement graph, and connected components are defined as usual. Unless stated otherwise, n and m will denote the number of vertices and edges, respectively, of the input graph.

2.2 A General Approach for the WED Problem

For a graph $G = (V, E)$ and a vertex $v \in V$, the *distance levels* with respect to v are

$$N_i(v) = \{w \in V \mid \text{dist}(v, w) = i\}$$

for all $i \in \mathbb{N}$. If v is fixed, we denote $N_i(v)$ by N_i . Let $R := V \setminus (\{v\} \cup N_1 \cup N_2)$, and let $G_v := G[N_2 \cup R]$ where vertices in N_2 get weight ∞ . Obviously, we have: G has a finite weight e.d. D_v with $v \in D_v$ if and only if G_v has a finite weight e.d. D , and $D_v = \{v\} \cup D$. In some cases, for every vertex $v \in V$, the WED problem can be efficiently solved on G_v , say in time $t(m)$ with $t(m) \geq m$.

If graph $G = (V, E)$ has an e.d. D then for any vertex $v \in V$, either $v \in D$ or one of its neighbors is in D . Thus, if $\deg_G(v) = \delta(G)$, one has to consider the WED problem on G_x for $\delta(G) + 1$ vertices $x \in N[v]$. Thus we obtain:

Lemma 1. *If for a graph class \mathcal{C} and input graph $G = (V, E)$ in \mathcal{C} , WED is solvable in time $t(m)$ on G_v for all $v \in V$ then WED is solvable in time $O(\delta(G) \cdot t(m))$ for graph class \mathcal{C} .*

2.3 Linear Forests

As already mentioned, if F is a linear forest such that one of its components contain $2P_3$, or two of its components contain P_3 , the WED problem is \mathbb{NP} -complete for F -free graphs.

For $2P_2$ -free graphs and more generally, for kP_2 -free graphs, it is known that the number of maximal independent sets is polynomial [1, 3, 6] and can be enumerated efficiently [8]. Since every e.d. is a maximal independent set, WED can be solved in polynomial time for kP_2 -free graphs.

In Section 3, we show that WED is solvable in polynomial time for $(P_5 + kP_2)$ -free graphs for every fixed k . Thus, the only remaining open case is the one of P_6 -free graphs; our approach used for $(P_5 + kP_2)$ -free graphs shows that if WED is polynomial for P_6 -free graphs then it is polynomial for $(P_6 + kP_2)$ -free graphs as well.

2.4 Modular Decomposition for the WED Problem

A set H of at least two vertices of a graph G is called *homogeneous* if $H \neq V(G)$ and every vertex outside H is either adjacent to all vertices in H , or to no vertex in H . Obviously, H is homogeneous in G if and only if H is homogeneous in the complement graph \overline{G} . A graph is *prime* if it contains no homogeneous set. A homogeneous set H is *maximal* if no other homogeneous set properly contains H . It is well known that in a connected graph G with connected complement \overline{G} , the maximal homogeneous sets are pairwise disjoint and can be determined in linear time using the so called *modular decomposition* (see, e.g., [4]). The *characteristic graph* G^* of G is the graph obtained from G by contracting each of the maximal homogeneous sets H of G to a single representative vertex $h \in H$, and connecting two such vertices by an edge if and only if they are adjacent in G . It is well known (and can be easily seen) that G^* is a prime graph.

For a disconnected graph G , the WED problem can be solved separately for each component. If \overline{G} is disconnected, then obviously, D is an e.d. of G if and only if D is a single universal vertex of G . Thus, from now on, we can assume that G and \overline{G} are connected, and thus, maximal homogeneous sets are pairwise disjoint. Obviously, we have:

Lemma 2. *Let H be a homogeneous set in G and D be an e.d. of G . Then the following properties hold:*

$$(i) \quad |D \cap H| \leq 1.$$

$$(ii) \quad \text{If } H \text{ has no vertex which is universal for } H \text{ then } |D \cap H| = 0.$$

Thus, the WED problem on a connected graph G for which \overline{G} is connected can be easily reduced to its characteristic graph G^* by contracting each homogeneous set H to a single representative vertex h whose weight is either ∞ if H has no universal vertex or the minimum weight of a universal vertex in H otherwise. Obviously, G has an e.d. D of finite weight if and only if G^* has a corresponding e.d. of the same weight. Thus, we obtain:

Theorem 1. *Let \mathcal{G} be a class of graphs and \mathcal{G}^* the class of all prime induced subgraphs of the graphs in \mathcal{G} . If the (W)ED problem can be solved for graphs in \mathcal{G}^* with n vertices and m edges in time $O(T(n, m))$, then the same problem can be solved for graphs in \mathcal{G} in time $O(T(n, m) + m)$.*

The modular decomposition approach leads to a linear time algorithm for WED on $2P_2$ -free graphs (see [2]) and to a very simple $O(\delta(G)m)$ time algorithm for WED on P_5 -free graphs (a simplified variant of the corresponding result in [2]); the modular decomposition approach is also described in [5].

3 WED in Polynomial Time for $(P_5 + kP_2)$ -Free Graphs

In this section we solve an open problem from [2]. Let G be a $(P_5 + P_2)$ -free graph and assume that G is not P_5 -free; otherwise, WED can be solved in time $O(\delta(G)m)$ as described in [2]. Let v_1, v_2, v_3, v_4, v_5 induce a P_5 H in G with edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5$, let $X = N(H) = \{x \mid x \notin V(H), \exists i(xv_i \in E)\}$ denote the neighborhood of H and let Y denote the anti-neighborhood $A(H)$ of H in G . Since G is $(P_5 + P_2)$ -free, we have:

Claim 1. *Y is an independent set.*

Assume that G has an e.d. D . Then:

Claim 2. $|V(H) \cup X \cap D| \leq 5$.

Proof of Claim 2. Obviously, $|V(H) \cap D| \leq 2$. If $|V(H) \cap D| = 2$ then $|X \cap D| = 0$ or $|X \cap D| = 1$ since D is an e.d. If $|V(H) \cap D| = 1$ then $|X \cap D| \leq 3$. Finally, if $|V(H) \cap D| = 0$ then $|X \cap D| \leq 5$ which shows Claim 2. \diamond

Let $D = D_1 \cup D_2$ be the partition of D into $D_1 = D \cap (V(H) \cup X)$ and $D_2 = D \cap Y$.

Claim 3. $D_2 = A(D_1) \cap Y$.

Proof of Claim 3. Since the anti-neighborhood Y of H is an independent set, every vertex in Y can only be dominated by itself or by a vertex from $D \cap X$. Thus, Claim 3 holds. \diamond

This leads to the following simple algorithm for checking whether G has an e.d. D :

1. Check whether G is P_5 -free; if yes, apply the corresponding algorithm for WED on P_5 -free graphs (which works in time $O(\delta(G)m)$), otherwise let H be a P_5 in G . Determine $X = N(H)$ and $Y = A(H)$. If Y is not independent then G is not $(P_5 + P_2)$ -free. Otherwise do the following:
2. For every independent set $S \subseteq V(H) \cup X$ with $|S| \leq 5$, check whether $S \cup (A(S) \cap Y)$ is an e.d.
3. If there is such a set then take one of minimum weight, otherwise output “ G has no e.d.”.

Obviously, the algorithm is correct and its running time is at most $O(n^5m)$.

For every fixed k , the approach for $(P_5 + P_2)$ -free graphs can be generalized to $(P_5 + kP_2)$ -free graphs: Assume inductively that WED can be solved in polynomial time for $(P_5 + (k-1)P_2)$ -free graphs. Thus, if the given graph G is $(P_5 + (k-1)P_2)$ -free, we can use the assumption,

otherwise find (in polynomial time) an induced subgraph H isomorphic to $P_5 + (k-1)P_2$ and determine its neighborhood X and its anti-neighborhood Y . Then similar claims as in the $(P_5 + P_2)$ -free case hold; in particular, Y is independent, $|(V(H) \cup X) \cap D| \leq 5 + 2k$ and we can check whether for such an independent set S and partition $D = D_1 \cup D_2$, $D_2 = A(D_1) \cap Y$ holds.

Corollary 1. *For every fixed k , WED is solvable in polynomial time for $(P_5 + kP_2)$ -free graphs.*

The approach can be easily generalized to $(H + kP_2)$ -free graphs whenever WED is solvable in polynomial time for H -free graphs. However, WED remains \mathbb{NP} -complete for $(H + kP_2)$ -free graphs whenever WED is \mathbb{NP} -complete for H -free graphs. If WED is solvable in polynomial time for P_6 -free graphs then it is solvable in polynomial time for $(P_6 + kP_2)$ -free graphs for every fixed k .

4 WED for $(P_4 + P_2)$ -Free Graphs in Time $O(\delta(G)m)$

In this section we slightly improve the time bound $O(nm)$ for WED [2] to $O(\delta(G)m)$ and simplify the proof in [2]. According to Lemma 1, for a vertex $v \in V$ with minimal degree $\delta(G)$, we check for all $x \in N[v]$ whether G_x has an e.d. D_x . We first collect some properties assuming that G is $(P_4 + P_2)$ -free and has an e.d. D_v . As before, let $G_v := G[N_2 \cup R]$; we can assume that G_v is prime. We are looking for an e.d. of G_v with finite weight and assume that $D_v \setminus \{v\}$ is such an e.d. Since G is $(P_4 + P_2)$ -free, we have:

Claim 4. $G[R]$ is a cograph.

Let R_1, \dots, R_ℓ denote the connected components of $G[R]$. Note that an e.d. of a connected cograph H has only one vertex, namely a universal vertex of H . Thus:

Claim 5. *For all $i \in \{1, \dots, \ell\}$, $|D_v \cap R_i| = 1$, and in particular, if $d \in D_v \cap R_i$ then d is universal for R_i .*

For all $i \in \{1, \dots, \ell\}$, let $D_v \cap R_i = \{d_i\}$. Let U_i be the set of universal vertices in R_i . Thus, if $U_i = \emptyset$ then G has no e.d., and if $U_i = \{d_i\}$ then necessarily $d_i \in D_v$. From now on, assume that for every $i \in \{1, \dots, \ell\}$, $|U_i| \geq 2$. We first claim that $\ell > 1$: Since G_v is prime and in case $\ell = 1$, G_v has an e.d. (of finite weight) if and only if G_v contains a universal vertex $z \in R_1$ for G_v , it follows:

Claim 6. *For prime G_v with e.d. D_v of finite weight, $\ell > 1$ holds.*

Since for finding an e.d. in G_v , every R_i can be reduced to the set U_i of its universal vertices (since the non-universal vertices in R_i cannot dominate all R_i vertices), we can assume that for all $i \in \{1, \dots, \ell\}$, R_i is a clique. If $|N_2| = 1$ then, since G_v is prime, for all $i \in \{1, \dots, \ell\}$, $|R_i| \leq 2$ and thus, G_v is a tree (in particular: If there are $i, j \in \{1, \dots, \ell\}$, $i \neq j$, $|R_i| = |R_j| = 1$ then G_v has no e.d., if there is exactly one $i \in \{1, \dots, \ell\}$ with $|R_i| = 1$ then this determines the D_v vertices in Z , and if for all $i \in \{1, \dots, \ell\}$, $|R_i| = 2$ then one has to choose the e.d. with smallest weight in the obvious way). From now on, let $|N_2| \geq 2$. If for all $z \in R$, either z has a join or a co-join to N_2 then N_2 would be homogeneous in G_v - contradiction. Thus, from now on we have:

Claim 7. *There is a vertex $z \in R$ having a neighbor and a non-neighbor in N_2 .*

Since G is $(P_4 + P_2)$ -free, we have:

Claim 8. *If $x \in N_2$ has a neighbor in R_i then for all $j \neq i$, it has at most one non-neighbor in R_j .*

In particular, this means:

Claim 9. *If $x \in N_2$ is adjacent to $d_i \in R_i \cap D_v$ then for all $j \neq i$, x has exactly one non-neighbor in Z_j which is the D_v -vertex in R_j .*

Claim 10. *If a vertex $z \in R_i$ has a non-neighbor $x \in N_2$ and $z \notin D_v$ then for all $j \neq i$, x has exactly one non-neighbor in R_j , namely $xd_j \notin E$ for $d_j \in R_j \cap D_v$.*

Proof of Claim 10. Let $z \in R_1$ have non-neighbor $x \in N_2$, and $z \notin D_v$, i.e., $z \neq d_1$. Then, since G is $(P_4 + P_2)$ -free, $xd_1 \in E$. By Claim 9, x has exactly one non-neighbor in R_j for each $j \in \{2, \dots, \ell\}$ (which is the corresponding D_v vertex in R_j). \square

Algorithm $(P_4 + P_2)$ -Free-WED- G_v :

Given: Graph $G = (V, E)$ and prime graph $G_v = G[N_2 \cup R]$ as constructed above with vertex weights $w(x)$; for all $x \in N_2$, $w(x) = \infty$.

Output: An e.d. D_v of G_v of finite minimum weight, if G_v has an e.d., or the statement that G is not $(P_4 + P_2)$ -free or G_v does not have any e.d. of finite weight.

- (0) Initially, $D_v := \emptyset$.
- (1) Check if $G[R]$ is a cograph. If not then G is not $(P_4 + P_2)$ -free - STOP. Else determine the connected components R_1, \dots, R_ℓ of $G[R]$. If $\ell = 1$ then G_v has no e.d. of finite weight - STOP.
- (2) For all $i \in \{1, \dots, \ell\}$, determine the set U_i of universal vertices in R_i . If for some i , $U_i = \emptyset$ then G_v has no e.d. - STOP. From now on, let $R_i := U_i$. If $U_i = \{d_i\}$ then $D_v := D_v \cup \{d_i\}$.
- (3) If $|N_2| = 1$ then check whether G_v is a tree and solve the problem in the obvious way. If $|N_2| > 1$, choose a vertex $z \in R$ with a neighbor $w \in N_2$ and a non-neighbor $x \in N_2$, say $z \in R_i$.
 - (3.1) Check if $z \in D_v$ leads to an e.d. (by using neighbor w of z and Claim 9).
 - (3.2) Check if $z \notin D_v$ leads to an e.d. (by using the non-neighbors of x in R_j , $j \neq i$ and Claim 10)
 - (3.3) If there is no e.d. in both cases (3.1) and (3.2) then either G is not $(P_4 + P_2)$ -free or has no e.d. of finite weight - STOP.

Theorem 2. *Algorithm $(P_4 + P_2)$ -Free-WED- G_v is correct and runs in time $O(m)$.*

Proof. *Correctness.* The correctness follows from Claims 4 - 10.

Time bound. The linear time bound is obvious. \square

Corollary 2. *WED is solvable in time $O(\delta(G)m)$ for $(P_4 + P_2)$ -free graphs.*

5 WED for Some Subclasses of P_6 -Free Graphs

Recall that the complexity of WED for P_6 -free graphs is open. In this section we consider WED for some subclasses of P_6 -free graphs. Let $G = (V, E)$ be a prime P_6 -free graph, let $v \in V$ and let N_1, N_2, \dots be the distance levels of v . Then we have:

$$N_k = \emptyset \text{ for all } k \geq 5 \text{ and } N_4 \text{ is an independent vertex set.} \quad (1)$$

Assume that G admits an e.d. D_v of finite weight with $v \in D_v$. Let $G_v := G[N_2 \cup N_3 \cup N_4]$; we can assume that G_v is prime. As before, $D_v \cap (N_1 \cup N_2) = \emptyset$; set $w(x) = \infty$ for $x \in N_2$. Thus, vertices of N_2 have to be dominated by vertices of $D_v \cap N_3$. We claim:

$$\text{At most one vertex in } D_v \cap N_3 \text{ has neighbors in } N_4. \quad (2)$$

Proof. Assume that there are two vertices $d_1, d_2 \in N_3 \cap D_v$ with neighbors in N_4 , say $x_i \in N_4$ with $d_i x_i \in E$ for $i = 1, 2$. Let $b_i \in N_2$ with $b_i d_i \in E$ for $i = 1, 2$. Since D_v is an e.d., $b_1 \neq b_2$ and $x_1 \neq x_2$ and d_1 misses b_2, x_2 while d_2 misses b_1, x_1 . Since N_4 is independent, $x_1 x_2 \notin E$ holds. Now, if $b_1 b_2 \in E$, then $x_1, d_1, b_1, b_2, d_2, x_2$ induce a P_6 in G , and if $b_1 b_2 \notin E$, there is a P_6 as well (together with N_1 vertices), a contradiction. \square

5.1 WED for $(P_6, S_{1,2,2})$ -free graphs in time $O(\delta(G)m)$

In this subsection we improve the time bound $O(n^2 m)$ for WED [2] to $O(\delta(G)m)$ and simplify the proof in [2]. Let $G = (V, E)$ be a connected $(P_6, S_{1,2,2})$ -free graph, let $v \in V$ and let N_1, N_2, \dots be the distance levels of v . We claim:

$$D_v \cap N_4 = \emptyset. \quad (3)$$

Proof. Assume to the contrary that there is a vertex $d \in D_v \cap N_4$. Let $c \in N_3$ be a neighbor of d , let $b \in N_2$ be a neighbor of c and let $a \in N_1$ be a neighbor of b . Then b has to be dominated by a D_v -vertex $d' \in N_3$, and since D_v is an e.d., $cd' \notin E$ and $dd' \notin E$ but now, v, a, b, c, d, d' induce an $S_{1,2,2}$, a contradiction. \square

Thus, set $w(x) := \infty$ for all $x \in N_4$. By (3), $D_v \subseteq N_3 \cup \{v\}$. Claim (2) means that D_v vertices in N_3 have either a join or a co-join to N_4 . Thus for finding an e.d. of G_v , we can delete all vertices in N_3 which have a neighbor and a non-neighbor in N_4 . Reducing G_v in this way gives G'_v ; again, we can assume that G'_v is prime. Now, N_4 is a module and thus, $|N_4| \leq 1$. Let $N_4 = \{z\}$ if N_4 is nonempty. Let Q_1, \dots, Q_ℓ denote the connected components of $G[N_3]$. We claim:

$$\text{No component } Q_i \text{ in } G[N_3] \text{ contains two vertices of } D_v. \quad (4)$$

Proof. Assume to the contrary that Q_1 contains $d_1, d_2 \in D_v$, $d_1 \neq d_2$. Let $x \in N_2$ be a neighbor of d_1 , and let P denote a path in Q_1 connecting d_1 and d_2 , i.e., either $P = (d_1, x_1, x_2, d_2)$ or $P = (d_1, x_1, x_2, x_3, d_2)$. Let a be a common neighbor of x and v . If $P = (d_1, x_1, x_2, d_2)$ then x is not adjacent to x_2 since G is $S_{1,2,2}$ -free (otherwise v, a, x, d_1, x_2, d_2 induce an $S_{1,2,2}$) and since G is P_6 -free, x is adjacent to x_1 (otherwise v, a, x, d_1, x_1, x_2 induce a P_6) but now v, a, x, x_1, x_2, d_2 induce a P_6 - contradiction. If $P = (d_1, x_1, x_2, x_3, d_2)$, the arguments are similar. \square

Thus, by (4), if D_v is an e.d. of G_v then $|D_v \cap Q_i| = 1$ for all i , $1 \leq i \leq \ell$, and the corresponding D_v -vertex is universal for Q_i . Thus, we can restrict Q_i to its universal vertices

U_i (which means that now, Q_i is a clique; if $U_i = \emptyset$ then G_v has no e.d.) In case $\ell = 1$ this means that if G_v has an e.d., G_v must have a universal vertex (since a D_v -vertex being universal for Q_1 must also be universal for $N_2 \cup N_4$) which is impossible since G_v is prime. This implies $\ell > 1$. If $|Q_i| = 1$ then the corresponding vertex in Q_i is a forced vertex for D_v and has to be added to D_v . We claim:

$$N_2 \text{ vertices cannot distinguish more than one } Q_i, i \in \{1, \dots, \ell\}. \quad (5)$$

Proof. Since G is $S_{1,2,2}$ -free, no vertex in N_2 can distinguish two components Q_i, Q_j in N_3 . In order to show (5), assume to the contrary that there are components Q_1, Q_2 in G'_v with $c_1, d_1 \in Q_1$ and $c_2, d_2 \in Q_2$ which are distinguished by vertices $x_1, x_2 \in N_2$ such that $x_1 d_1 \in E$, $x_1 c_1 \notin E$, and $x_2 d_2 \in E$, $x_2 c_2 \notin E$. Since no vertex in N_2 can distinguish two components Q_i, Q_j , $x_1 \neq x_2$ holds, and since G is $S_{1,2,2}$ -free, $x_1 c_2 \notin E$ and $x_1 d_2 \notin E$, and by symmetry also $x_2 c_1 \notin E$ and $x_2 d_1 \notin E$, but now $c_1, d_1, x_1, x_2, d_2, c_2$ induce a P_6 if $x_1 x_2 \in E$ or a P_6 together with N_1 vertices if $x_1 x_2 \notin E$ - contradiction. \square

First assume $N_4 \neq \emptyset$, i.e., $N_4 = \{z\}$. For every $i \in \{1, \dots, \ell\}$, let Q_i^+ denote the neighbors of z in Q_i and let Q_i^- denote the non-neighbors of z in Q_i . By (5), at most one Q_i has more than two vertices, say $|Q_i| \leq 2$ for all $i \in \{2, \dots, \ell\}$ since in this case, Q_i^+ and Q_i^- are modules. Since D_v is an e.d., there is a vertex $d \in D_v$ with $dz \in E$; say $d \in Q_i^+$. Let $b \in N_2$ with $bd \in E$ and $a \in N_1$ with $ab \in E$. Now for $j \neq i$, every neighbor $x \in Q_j^+$ of z must see b since otherwise v, a, b, d, z, x induce a P_6 , and every non-neighbor $y \in Q_j^-$ of z must miss b since otherwise v, a, b, d, z, y induce an $S_{1,2,2}$ but if Q_j contains both x and y then v, a, b, d, x, y induce an $S_{1,2,2}$ - contradiction. Thus, we have:

$$\text{At most one } Q_i \text{ has more than one vertex.} \quad (6)$$

Say $|Q_i| = 1$ for all $i \in \{2, \dots, \ell\}$. If $N_4 = \emptyset$, this holds as well.

This leads to the following algorithm for WED with time bound $O(m)$ for every v :

Algorithm $(P_6, S_{1,2,2})$ -Free-WED- G_v :

Given: Connected graph $G = (V, E)$ and prime graph $G_v = G[N_2 \cup N_3 \cup N_4]$ as constructed above with vertex weights $w(x)$; for all $x \in N_2 \cup N_4$, $w(x) = \infty$.

Output: An e.d. D_v of G_v of finite weight, if G_v has such an e.d., or the statement that G is not $(P_6, S_{1,2,2})$ -free or G_v does not have any e.d. of finite weight.

- (0) Initially, $D_v := \emptyset$.
- (1) Check if $G[N_5] = \emptyset$; if not then G is not P_6 -free - STOP. Else determine the connected components Q_1, \dots, Q_ℓ of $G[N_3]$. If $\ell = 1$ then G_v has no e.d. of finite weight - STOP.
- (2) For all $i \in \{1, \dots, \ell\}$, determine the set U_i of universal vertices in Q_i . If $U_i = \emptyset$ then G_v has no e.d. - STOP. From now on, let $Q_i := U_i$. If $U_i = \{d_i\}$ then $D_v := D_v \cup \{d_i\}$. Delete all vertices $x \in N_3$ which have a neighbor and a non-neighbor in N_4 . Contract N_4 to one vertex z if $N_4 \neq \emptyset$.
- (3) For all $|Q_i| = 1$, add its vertex to D_v and delete its neighbors from N_2 . If there is an $i \in \{1, \dots, \ell\}$ with $|Q_i| > 1$, say $|Q_1| > 1$, then check whether there is a vertex $d \in Q_1$ which has exactly the remaining N_2 vertices as its neighborhood in N_2 and sees z for $N_4 = \{z\}$.

- (4) Finally check whether D_v is an e.d. of G_v - if not then either G is not $(P_6, S_{1,2,2})$ -free or has no e.d. (containing v) of finite weight.

Theorem 3. *Algorithm $(P_6, S_{1,2,2})$ -Free-WED- G_v is correct and runs in time $O(m)$.*

Proof. *Correctness.* The correctness follows from the previous claims and considerations.

Time bound. The linear time bound is obvious. \square

Corollary 3. *WED is solvable in time $O(\delta(G)m)$ for $(P_6, S_{1,2,2})$ -free graphs.*

5.2 WED for P_6 -free graphs of diameter 3

In this subsection, we reduce the WED problem on P_6 -free graphs in polynomial time to such graphs having diameter 3. Let D be an e.d. of G . By Theorem 1, we can assume that G is prime. As before, we check for every vertex $v \in V$ if $v \in D$ leads to an e.d. of G . For this purpose, let N_i , $i \geq 1$, again be the distance levels of v . Recall that by (1), $N_k = \emptyset$ for $k \geq 5$ and N_4 is an independent vertex set, and by (2), at most one vertex in $D_v \cap N_3$ has neighbors in N_4 .

Recall that $A(x)$ denotes the anti-neighborhood of x . Thus, if $N_4 \neq \emptyset$ then check for every vertex $x \in N_3$ whether $\{v, x\} \cup (A(x) \cap N_4)$ is an e.d. in G ; since N_4 is independent, vertices in N_4 not dominated by x must be in D_v . This can be done in polynomial time for all v with $N_4 \neq \emptyset$.

Now we can assume that the diameter of G is at most 3, i.e., for every $v \in V$, the distance level N_4 is empty.

Corollary 4. *If WED is solvable in polynomial time for P_6 -free graphs of diameter 3 then WED is solvable in polynomial time for P_6 -free graphs.*

6 WED for $(2P_3, S_{1,2,2})$ -Free Graphs in Time $O(\delta(G)n^3)$

In this section we improve the time bound $O(n^5)$ for WED [2] to $O(\delta(G)n^3)$ and simplify the proof in [2]. Let $G = (V, E)$ be a connected $(2P_3, S_{1,2,2})$ -free graph, let $v \in V$ and let N_1, N_2, \dots be the distance levels of v . Since G is $2P_3$ -free, we have $N_k = \emptyset$ for $k \geq 6$. Let $R := V \setminus (\{v\} \cup N_1 \cup N_2)$. Assume that G admits an e.d. D_v of finite weight with $v \in D_v$. Let $G_v := G[N_2 \cup N_3 \cup N_4 \cup N_5]$, i.e. $G_v = G[N_2 \cup R]$; we can assume that G_v is prime. Since D_v is an e.d., $R \neq \emptyset$. Let Q_1, \dots, Q_ℓ , $\ell \geq 1$, denote the connected components of $G[R]$. Clearly, $D_v \cap Q_i \neq \emptyset$ for every i . Let $D_v \setminus \{v\} = \{d_1, \dots, d_k\}$, and assume that $k \geq 2$ (otherwise, G_v would have a universal vertex which is impossible for a prime graph). Since G is $S_{1,2,2}$ -free and D_v is an e.d., we have:

$$\text{Every } x \in N_2 \text{ seeing a vertex } d_i \in D_v \text{ misses } N[d_j] \cap R, j \neq i. \quad (7)$$

We claim:

$$\text{For every } i = 1, \dots, k, N[d_i] \cap R \text{ is a clique.} \quad (8)$$

Proof. Suppose that $N[d_1] \cap R$ is not a clique, i.e., there are neighbors $x, y \in R$ of d_1 with $xy \notin E$. Let $b \in N_2$ be a neighbor of d_2 . By (7), b misses x and y but now, a, b, d_2, x, d_1, y induce $2P_3$, a contradiction. \square

Next we claim:

$$\text{If } k \geq 3 \text{ then } G[N_3] \text{ is the disjoint union of cliques.} \quad (9)$$

Proof. Suppose that $k \geq 3$ and there is an edge $uw \in E$ for $u \in N(d_2) \cap R$ and $w \in N(d_3) \cap R$. Let $x \in N_2$ with $xd_1 \in E$ and $a \in N_1$ with $ax \in E$. Then by (7), v, a, x, d_2, u, w induce $2P_3$, a contradiction. \square

Thus, for $k \geq 3$, every Q_i is a clique containing exactly one D_v vertex:

$$|D_v \cap Q_i| = 1. \quad (10)$$

If Q_i is a single vertex q_i then q_i is forced and has to be added to D_v . From now on assume that for all i , $|Q_i| \geq 2$. By (7), we have:

$$\text{If } z \in N_2 \text{ sees } Q_i \text{ then it misses all } Q_j, j \neq i. \quad (11)$$

Let S_i denote the set of vertices in N_2 distinguishing vertices in Q_i . Since Q_i is not a module, $S_i \neq \emptyset$ for all $i \in \{1, \dots, k\}$. Let U_i denote the vertices in Q_i which have a join to S_i ; $U_i \neq \emptyset$ since S_i vertices must have a D_v neighbor in Q_i . We claim:

$$\text{For all } i \in \{1, \dots, k\}, |U_i| = 1. \quad (12)$$

Proof. Assume to the contrary that $|U_1| > 1$. If $x \in S_1$ then by (11), $xd_1 \in E$, i.e., $d_1 \in U_1$. Now, a vertex distinguishing U_1 would be in S_1 but vertices in U_1 have a join to S_1 and thus cannot be distinguished which is a contradiction to the assumption that G_v is prime. \square

The other case when $k \leq 2$, i.e., $|D_v \setminus \{v\}| \leq 2$, can be easily done via the adjacency matrix of G : For any pair $x, y \in R$, $x \neq y$, with $xy \notin E$, check whether all other vertices in G_v are adjacent to exactly one of them; this can be done in time $O(n^3)$.

This leads to the following:

Algorithm $(2P_3, S_{1,2,2})$ -Free-WED- G_v :

Given: Connected graph $G = (V, E)$ and prime graph $G_v = G[N_2 \cup R]$ as constructed above with vertex weights $w(x)$; for all $x \in N_2$, $w(x) = \infty$.

Output: An e.d. D_v of G_v of finite weight if G_v has such an e.d., or the statement that G is not $(2P_3, S_{1,2,2})$ -free or G_v does not have any e.d. of finite weight.

- (0) Initially, $D_v := \emptyset$.
- (1) Determine N_1, N_2 and R . If $R = \emptyset$ then $G_v = G[N_2 \cup R]$ has no e.d. - STOP. Else determine the connected components Q_1, \dots, Q_ℓ of R .
- (2) If $G[R]$ is not the disjoint union of cliques Q_1, \dots, Q_ℓ , $\ell \geq 3$, then check whether G_v has a finite weight e.d. with two vertices, and determine an e.d. with minimum weight. If not, G_v has no e.d. of finite weight - STOP.
- (3) (Now $G[R]$ is the disjoint union of cliques Q_1, \dots, Q_ℓ , $\ell \geq 3$) If $Q_i = \{d_i\}$ then d_i is forced - $D_v := D_v \cup \{d_i\}$. If $|Q_i| > 1$ then determine the set S_i of vertices distinguishing Q_i , and determine the set U_i of vertices in Q_i having a join to S_i . If $U_i = \emptyset$ then G_v has no e.d. - STOP. Otherwise, $U_i = \{d_i\}$ and d_i is forced - $D_v := D_v \cup \{d_i\}$.

- (4) Finally check whether D_v is an e.d. of finite weight of G_v - if not then either G is not $(2P_3, S_{1,2,2})$ -free or has no e.d. of finite weight.

Theorem 4. *Algorithm $(2P_3, S_{1,2,2})$ -Free-WED- G_v is correct and runs in time $O(n^3)$.*

Proof. *Correctness.* The correctness follows from the previous claims and considerations.

Time bound. The time bound is obvious since step (1) can be done in time $O(m)$, step (2) can be done in time $O(n^3)$, and steps (3) and (4) can be done in time $O(m)$. \square

Corollary 5. *WED is solvable in time $O(\delta(G)n^3)$ for $(2P_3, S_{1,2,2})$ -free graphs.*

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